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# Casimir effect at finite temperature

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**Abstract.** The high-temperature expansion of the grand thermodynamic potential of a non-conformally invariant spin-0 gas in an arbitrary ultrastatic spacetime with boundary is given in terms of the Minakshisundaram–Pleijel coefficients of the heat-kernel and the zeta function of the spatial section. The general formula is then used to find the expansion in the case of a massive bosonic field subject to Dirichlet boundary conditions on hyper-cuboids in a flat $n$-dimensional spacetime. A detailed analysis of inhomogeneous multi-dimensional Epstein zeta functions is necessary and some new properties of them are derived. Finally the thermodynamics of the system is considered.

## 1. Introduction

The Casimir effect [1] is one of the most beautiful and simple manifestations of the striking vacuum structure of quantum field theory (for general reviews see [2, 3]). The effect is simply due to the influence of the boundary conditions imposed on the vacuum configuration. The dependence on the boundary conditions is made explicit by the formal definition of the Casimir energy

$$E_c[\partial\mathcal{M}] = E_0[\partial\mathcal{M}] - E_0[0] \tag{1.1}$$

where $E_0[\partial\mathcal{M}]$ is the zero-point energy in the presence of a boundary $\partial\mathcal{M}$ and $E_0[0]$ the zero-point energy without boundary.

This definition is only meaningful, when supplemented with a regularization method leading to finite results. Zeta function regularization techniques of recent years have been shown to provide a very powerful method in calculating quantities like the vacuum energy or the thermodynamic potential at high temperature [4–16]. Based on new results concerning the interchangeability of the order of summations [7] the Casimir energy (1.1) was obtained for many situations.

Up to now, most of the calculations concerning the Casimir energy were done at temperature $T=0$. The aim of this paper is to develop a systematic approach to find the Casimir energy at $T>0$. The appropriate generalization of the Casimir energy (1.1) to finite temperature will be the difference between the free energy in the presence and in the absence of constraints [2]

$$\psi_c[\partial\mathcal{M}] = \psi[\partial\mathcal{M}] - \psi[0]. \tag{1.2}$$

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Using zeta function regularization, Dowker and Kennedy [17] derived a high temperature expansion of the free energy of a massless spin-0 gas in an arbitrary ultrastatic spacetime. These considerations were generalized to the thermodynamic potential of a non-conformally invariant gas in a static spacetime without boundary [18, 19]. A non-vanishing chemical potential has been introduced in [19] without much additional effort. Specializing to the flat-space limit, the results were found to agree with [20].

In order to investigate the Casimir energy (1.2) for a non-conformally invariant gas at high temperature, it is necessary to generalize the expansion of [19] to a spacetime where the spatial section has a boundary. This is done in section 2. We restrict ourselves to an ultrastatic spacetime, because this is all we will need in the following, but the expressions may also be given in an arbitrary static spacetime with boundary using the techniques described in [18, 19, 21, 22].

The expansion is given in terms of the Minakshisundaram–Pleijel coefficients of the heat-kernel and the zeta function of the spatial section. Obtaining the high-temperature expansion in a special spacetime with specific boundary conditions imposed on the field, essentially means finding the corresponding Minakshisundaram–Pleijel coefficients and analysing the zeta function of the spatial section. It is one point of this paper to show that preceeding in this way is a powerful method in calculating high-temperature expansions for special cases.

To illustrate this, the thermodynamic potential for a massive bosonic field subject to Dirichlet boundary conditions on hypercuboids of arbitrary dimensions is derived, so extending some known results [14, 23–25] to a non-zero chemical potential and to a field subject to external conditions. In section 3 the Minakshisundaram–Pleijel coefficients resulting from the described problem are calculated using Poisson resummation. In section 4 we analyse the zeta function of the spatial section. It is seen that a detailed discussion of the inhomogeneous multidimensional Epstein zeta function is necessary. This is done in section 5 and we obtain some interesting properties of them. Using the results of sections 2–5 the high-temperature expansion of the thermodynamic potential including non-zero chemical potential is given for a field subject to the described external conditions in section 6. The results of two parallel plates in three spatial dimensions is easily extracted. In section 7 the thermodynamics of the system is analysed.

Other boundary conditions can be treated by slightly modifying the given analysis.

## 2. High-temperature expansion in an ultrastatic spacetime

We shall first concern ourselves with the finite-temperature behaviour of a field theory in the $n$-dimensional ultrastatic spacetime

$$ds^2 = d\tau^2 + h_{ij}(x)\, dx^i\, dx^j \tag{2.1}$$

described by the field equation

$$\left\{\left(\frac{\partial}{\partial\tau} - \mu\right)^2 + \Delta - \xi R - m^2\right\}\phi = 0 \tag{2.2}$$

with some, for the moment, unspecified boundary condition imposed on the field $\phi$ on the boundary $\partial\mathcal{M}$ of the spatial section $\mathcal{M}$. In this Euclidean formulation of the field theory the chemical potential $\mu$ has been incorporated as described, for example, by Actor [14], $\tau$ is the imaginary time compactified to a circle of size $\beta$, $\beta$ is the inverse temperature, and $\Delta = |h|^{-1/2}(\partial/\partial x^i)(|h|^{1/2}h^{ij}\partial/\partial x^j)$ is the Laplacian of the spatial section.

The finite thermodynamic potential is defined by [26]

$$\psi[\beta, \mu] = \frac{1}{\beta}\{\zeta_n(0, \beta, \mu)\ln\lambda^2 - \zeta_n'(0, \beta, \mu)\} \tag{2.3}$$

where $\lambda$ is the scaling length, the prime denotes differentiation with respect to $s$ and $\zeta_n(s, \beta, \mu)$ is the zeta function associated with the operator

$$D = -\left(\frac{\partial}{\partial\tau} - \mu\right)^2 - \Delta + \xi R + m^2. \tag{2.4}$$

That means

$$\zeta_n(s, \beta, \mu) = \sum_m \nu_m^{-s} = \frac{1}{\Gamma(s)}\sum_m \int_0^\infty \mathrm{d}t\, t^{s-1}\exp(-\nu_m t) \tag{2.5}$$

valid for $\mathrm{Re}\, s > n/2$, with

$$Du_m = \nu_m u_m. \tag{2.6}$$

In an ultrastatic manifold time is completely separated from space and using

$$u_{l,k} = \frac{1}{\beta}\exp\left(\frac{2\pi i l}{\beta}\tau\right)g_k(\boldsymbol{x}) \tag{2.7}$$

the eigenvalues $\nu_m$ may be written in the form

$$\nu_{l,k} = -\left(\frac{2\pi i l}{\beta} - \mu\right)^2 + \lambda_k \qquad l \in \mathbb{Z} \tag{2.8}$$

where the $\lambda_k$ are the eigenvalues of the operator $-\Delta + \xi R + m^2$ with some given boundary condition imposed on $g_k(\boldsymbol{x})$.

The high-temperature expansion of the zeta function (2.5) is obtained by using the short-time asymptotic expansion of the integrated proper-time propagator [27–29]

$$K(t) = \sum_k \exp(-\lambda_k t) \sim \frac{1}{(4\pi t)^{(n-1)/2}} \sum_{j=0,\frac{1}{2},1,\ldots} c_j t^j. \tag{2.9}$$

Separating off the $l = 0$ term and neglecting exponentially small terms for $t \to 0$ [17–19] one finds

$$\zeta_n(s, \beta, \mu) = \zeta_{n-1}(s, \mu) + \frac{2}{(4\pi)^{(n-1)/2}\Gamma(s)} \sum_{j=0,\frac{1}{2},1,\ldots}^{\infty} c_j \Gamma\left(\frac{z}{2}\right) b^z \,\mathrm{Re}\, \zeta_{\mathrm{H}}(z, 1 + i\rho) \tag{2.10}$$

with $z = 2s + 2j - n + 1$, $b = \beta/2\pi$, $\rho = \beta\mu/2\pi$, $\zeta_{\mathrm{H}}(s, w)$ is the Hurwitz zeta function and $\zeta_{n-1}(s, \mu)$ is the zeta function on the spatial section,

$$\zeta_{n-1}(s, \mu) = \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}t\, t^{s-1}\exp(\mu^2 t)K(t) \qquad \mathrm{Re}\, s > \frac{n-1}{2} \tag{2.11}$$

(where for convergence $\mu^2$ must be smaller than the smallest eigenvalue $\lambda_k$). Expanding $\zeta_n(s, \beta, \mu)$ around $s = 0$ [19, 30], the thermodynamic potential (2.3) may be written in the form

$$\psi[\beta,\mu]=-\frac{1}{\beta}\zeta'_{n-1}(0,\mu)+\frac{1}{(4\pi)^{n/2}}\left\{2c_{n/2}\left[\mathrm{Re}\,\psi(1+\mathrm{i}\rho)-\ln\left(\frac{\beta}{4\pi\lambda}\right)\right]\right.$$
$$+\frac{4\sqrt{\pi}}{\beta}c_{(n-1)/2}\ln\left(\frac{\beta\sinh\pi\rho}{\pi\rho}\right)$$
$$+\sqrt{4\pi}\sum_{r=1}^{[(n-1)/2]}c_{(n-1)/2-r}\left[-\frac{2(-1)^r\,\mathrm{Re}\,\zeta'_{\mathrm{H}}(-2r,1+\mathrm{i}\rho)}{\pi r!}\left(\frac{\beta}{2\pi}\right)^{-2r-1}\right.$$
$$\left.+\frac{\mu^{2r}}{\beta r!}\left(\gamma+2\ln\left(\frac{\beta}{2\pi}\right)+\psi(1+r)\right)\right]$$
$$+\sum_{r=1}^{[n/2]}c_{n/2-r}\left(\frac{\beta}{2\pi}\right)^{-2r}\frac{\sqrt{\pi}(-1)^r}{r\Gamma(r+\frac{1}{2})}\mathrm{Re}\,B_{2r}(\mathrm{i}\rho)$$
$$\left.+\sqrt{4\pi}\sum_{r=\frac{1}{2},1,\ldots}^{\infty}c_{n/2+r}\left(\frac{\beta}{2\pi}\right)^{2r}\frac{(-1)^{2r}\Gamma\left(\frac{2r+1}{2}\right)\mathrm{Re}\,\psi^{(2r)}(1+\mathrm{i}\rho)}{\pi(2r)!}\right\} \tag{2.12}$$

where $\psi(z)=\Gamma'(z)/\Gamma(z)$ and $B_r$ are the Bernoulli polynomials.

This is the generalization of the high-temperature expansion of the thermodynamic potential given in [19] to a spatial section with boundary. Setting $\mu=0$, the high-temperature expansion of the free energy is found ([18], equation (9)).

In my opinion, equation (2.12) is a very suitable starting point in calculating high-temperature expansions for specific configurations. The remaining thing to do is to find the coefficients $c_j$ of the heat-kernel (2.9) and to consider the zeta function (2.11) on the spatial section. This is demonstrated in the next sections by treating a previously incompletely solved problem.

## 3. Example

We now restrict the ultrastatic spacetime to have an Euclidean spatial section. The field is supposed to obey a mixture of Dirichlet and periodic boundary conditions, i.e. to vanish on $p$ perpendicular pairs of parallel hyperplanes or plates held at distances $L_1,\ldots,L_p$ and to be periodic with periodicity length $L$ tending to infinity in $q$ directions, with $p+q=n-1$.

In the notation of section 2 one has

$$\lambda_{l_1\ldots l_q n_1\ldots n_p}=m^2+\sum_{i=1}^{q}\left(\frac{2\pi l_i}{L}\right)^2+\sum_{i=1}^{p}\left(\frac{\pi n_i}{L_i}\right)^2 \qquad l_i\in\mathbb{Z},\ n_i\in\mathbb{N}. \tag{3.1}$$

Using equation (2.12) the problem of the determination of the high-temperature expansion of the resulting thermodynamic potential thus reduces almost to the calculation of the asymptotic expansion for $t\to 0$ of the heat-kernel $K(t)$, (2.9), and $\zeta'_{n-1}(0,\mu)$, (2.11), where $\lambda_k$ is given by (3.1). Let us start with the determination of the Minakshisundaram–Pleijel coefficients $c_j$; the examination of the zeta function of the spatial section will be done in sections 4 and 5.

Taking into consideration only the leading order in $L$ (this means the propagation of the field is taken to be free in $q$ directions) we are looking for the asymptotic expansion for $t \to 0$ of

$$K(t) = \left(\frac{L}{2\pi}\right)^q \sum_{n_1,\ldots,n_p=1}^{\infty} \int \mathrm{d}^q k \exp\left\{-\left[k^2 + \left(\frac{\pi n_1}{L_1}\right)^2 + \ldots + \left(\frac{\pi n_p}{L_p}\right)^2 + m^2\right]t\right\}$$
$$= \frac{L^q}{(4\pi t)^{q/2}} \sum_{n_1,\ldots,n_p=1}^{\infty} \exp\left\{-\left[\left(\frac{\pi n_1}{L_1}\right)^2 + \ldots + \left(\frac{\pi n_p}{L_p}\right)^2 + m^2\right]t\right\}. \tag{3.2}$$

This may be found completely by the method of Poisson resummation [31]

$$\sum_{n=-\infty}^{\infty} \exp(-xn^2) = \left(\frac{\pi}{x}\right)^{1/2} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{\pi^2 n^2}{x}\right). \tag{3.3}$$

Using equation (3.3), one first finds

$$\sum_{n=1}^{\infty} \exp(-an^2 t) = \frac{1}{2}\left\{\left(\frac{\pi}{ta}\right)^{1/2} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{\pi^2 n^2}{at}\right) - 1\right\}$$
$$= \frac{1}{2}\left\{\left(\frac{\pi}{ta}\right)^{1/2} - 1 + \mathrm{O}\left(\exp\left[-\frac{\pi^2}{at}\right]\right)\right\} \tag{3.4}$$

for $t \to 0$. Looking at equation (2.5) one sees that the exponentially damped terms for $t \to 0$ in (3.4) lead to exponentially damped terms of $\zeta_n(s, \beta, \mu)$ for $\beta \to 0$, this means $T \to \infty$. So it is reasonable to neglect these terms in the expansion we are looking for and we will ignore them from now on. So $K(t)$ is readily seen to be

$$K(t) = \left(\frac{1}{4\pi t}\right)^{(n-1)/2} L^q \exp(-tm^2) \sum_{l=0,\frac{1}{2},1,\ldots}^{p/2} (-1)^{2l} \pi^l A(p-2l) t^l \tag{3.5}$$

where

$$A(l) = \sum_{\{i_1,\ldots,i_l\}} L_{i_1} \ldots L_{i_l}. \tag{3.6}$$

$\Sigma_{\{i_1,\ldots,i_l\}}$ denotes the sum over all possible choices of the $i_1 < i_2 < \ldots < i_l$ among $1, \ldots, p$, and $A(0) = 1$.

Expanding $\exp(-m^2 t)$ in its Taylor series it is seen furthermore that

$$K(t) = \left(\frac{1}{4\pi t}\right)^{(n-1)/2} L^q \sum_{l=0,\frac{1}{2},1,\ldots}^{\infty} t^l (-1)^{2l} \left\{\sum_{k=0}^{[l]} \frac{(-1)^k m^{2k} \pi^{l-k}}{k!} A(p+2k-2l)\right\} \tag{3.7}$$

where $A(l) = 0$ for $l < 0$ is used.

So the Minakshisundaram–Pleijel coefficients are given by

$$c_l = L^q (-1)^{2l} \sum_{k=0}^{[l]} \frac{(-1)^k m^{2k} \pi^{l-k}}{k!} A(p+2k-2l). \tag{3.8}$$

These are the coefficients which have to be used in equation (2.12). To obtain the full expansion, only $\zeta_{n-1}(s, \mu)$ has to be considered for the specific example.

## 4. Zeta function of the spatial section

This last step will lead us to consider inhomogeneous multidimensional Epstein zeta functions. In contrast to the intensive studies concerning homogeneous Epstein zeta functions (e.g. [5-11] and references cited therein), not so much is known about the inhomogeneous ones [7, 10, 11, 32-35]. So a more detailed consideration is necessary.

Using equation (2.11) together with (3.2), it is seen that

$$\zeta_{n-1}(s,\mu)=\frac{L^q}{(4\pi)^{q/2}\Gamma(s)}\sum_{n_1,\ldots,n_p=1}^{\infty}\int_0^{\infty}\mathrm{d}t\, t^{s-1-q/2}$$
$$\times\exp\left\{-\left[\left(\frac{\pi n_1}{L_1}\right)^2+\ldots+\left(\frac{\pi n_p}{L_p}\right)^2+m^2-\mu^2\right]t\right\}$$
$$=\frac{L^q}{(4\pi)^{q/2}}\frac{\Gamma(s-q/2)}{\Gamma(s)}E_p^{m^2-\mu^2}\left(s-\frac{q}{2};\left(\frac{\pi}{L_1}\right)^2,\ldots,\left(\frac{\pi}{L_p}\right)^2\right) \qquad (4.1)$$

where the inhomogeneous Epstein zeta function

$$E_p^{c^2}(\nu;a_1,\ldots,a_p)=\sum_{n_1,\ldots,n_p=1}^{\infty}[a_1n_1^2+\ldots+a_pn_p^2+c^2]^{-\nu} \qquad (4.2)$$

$a_1,\ldots,a_p>0$, $c\in\mathbb{R}$, valid for $\mathrm{Re}\,\nu>p/2$, has been introduced (we assume $|\mu|<m$ for the moment). For determining $\zeta'_{n-1}(0,\mu)$ it is necessary to expand the RHS of equation (4.1) around $s=0$. Already at this point $E_p^{c^2}(\nu;a_1,\ldots,a_p)$ may be seen to have poles of order 1 at $\nu=-M-\frac{1}{2}$, $M\in\mathbb{N}_0$. Let us explain this shortly.

General zeta function theory (see e.g. [36-39]) tells us that [19]:

$$\zeta_{n-1}(0,\mu)=\frac{1}{(4\pi)^{(n-1)/2}}\sum_{k=0}^{[(n-1)/2]}\frac{\mu^{2k}}{k!}c_{(n-1)/2-k} \qquad (4.3)$$

where in our calculation the coefficients $c_{(n-1)/2-k}$ are given by equation (3.8). All the coefficients $c_l$ are non-vanishing, i.e. $\zeta_{n-1}(0,\mu)$ is non-vanishing. In order that this is fulfilled, the Epstein zeta function in equation (4.1) must have poles for odd $q=2M+1$, $M\in\mathbb{N}$, at $s=0$, because then the pole of the gamma function at $s=0$ and the pole of the Epstein zeta function at $-q/2$ cancel and a non-vanishing remainder is obtained. The relevant expansion in equation (4.1) for $s\to 0$ thus reads

$$E_p^{c^2}(s-M-\tfrac{1}{2};a_1,\ldots,a_p)=\frac{1}{s}\mathrm{Res}\,E_p^{c^2}(-M-\tfrac{1}{2};a_1,\ldots,a_p)$$
$$+O_p^{c^2}(-M-\tfrac{1}{2};a_1,\ldots,a_p)+\mathrm{O}(s). \qquad (4.4)$$

For $q=2M$ even, the poles of the gamma function in (4.1) at $s=0$ cancel and the expansion needed for $\zeta'_{n-1}(0,\mu)$ is then

$$E_p^{c^2}(s-M;a_1,\ldots,a_p)=E_p^{c^2}(-M;a_1,\ldots,a_p)+sE_p'^{c^2}(-M;a_1,\ldots,a_p)+\mathrm{O}(s^2). \qquad (4.5)$$

Expansions (4.4) and (4.5) will be given explicitly as a function of the $a_i$ in section 5.

Thus $\zeta_{n-1}(s,\mu)$ can be written in the form

$$\zeta_{n-1}(s,\mu)=\frac{L^q}{(4\pi)^{q/2}}\frac{(-1)^{q/2}}{(q/2)!}\left\{E_p^{m^2-\mu^2}\left(-\frac{q}{2};\left(\frac{\pi}{L_1}\right)^2,\ldots,\left(\frac{\pi}{L_p}\right)^2\right)\right.$$
$$+s\left[E_p^{m^2-\mu^2}\left(-\frac{q}{2};\left(\frac{\pi}{L_1}\right)^2,\ldots,\left(\frac{\pi}{L_p}\right)^2\right)\left[\gamma+\psi\left(\frac{q}{2}+1\right)\right]\right.$$
$$\left.\left.+E_p'^{m^2-\mu^2}\left(-\frac{q}{2};\left(\frac{\pi}{L_1}\right)^2,\ldots,\left(\frac{\pi}{L_p}\right)^2\right)\right]\right\}+\mathrm{O}(s^2) \qquad \text{for } q \text{ even} \qquad (4.6)$$

and

$$\zeta_{n-1}(s,\mu)=\frac{L^q}{(4\pi)^{q/2}}\Gamma\left(-\frac{q}{2}\right)\left\{\operatorname{Res} E_p^{m^2-\mu^2}\left(-\frac{q}{2};\left(\frac{\pi}{L_1}\right)^2,\ldots,\left(\frac{\pi}{L_p}\right)^2\right)\right.$$
$$+s\left[\operatorname{Res} E_p^{m^2-\mu^2}\left(-\frac{q}{2};\left(\frac{\pi}{L_1}\right)^2,\ldots,\left(\frac{\pi}{L_p}\right)^2\right)\left[\gamma+\psi\left(\frac{q}{2}+1\right)\right]\right.$$
$$\left.\left.+O_p^{m^2-\mu^2}\left(-\frac{q}{2};\left(\frac{\pi}{L_1}\right)^2,\ldots,\left(\frac{\pi}{L_p}\right)^2\right)\right]\right\}+O(s^2) \qquad \text{for } q \text{ odd.} \tag{4.7}$$

Differentiating with respect to $s$ one finds

$$\zeta'_{n-1}(0,\mu)=\frac{L^q}{(4\pi)^{q/2}}\frac{(-1)^{q/2}}{(q/2)!}\left\{E_p'^{m^2-\mu^2}\left(-\frac{q}{2};\left(\frac{\pi}{L_1}\right)^2,\ldots,\left(\frac{\pi}{L_p}\right)^2\right)\right.$$
$$\left.+E_p^{m^2-\mu^2}\left(-\frac{q}{2};\left(\frac{\pi}{L_1}\right)^2,\ldots,\left(\frac{\pi}{L_p}\right)^2\right)\left[\gamma+\psi\left(\frac{q}{2}+1\right)\right]\right\} \qquad \text{for } q \text{ even} \tag{4.8}$$

and

$$\zeta'_{n-1}(0,\mu)=\frac{L^q}{(4\pi)^{q/2}}\Gamma\left(-\frac{q}{2}\right)\left\{O_p^{m^2-\mu^2}\left(-\frac{q}{2};\left(\frac{\pi}{L_1}\right)^2,\ldots,\left(\frac{\pi}{L_p}\right)^2\right)\right.$$
$$+\operatorname{Res} E_p^{m^2-\mu^2}\left(-\frac{q}{2};\left(\frac{\pi}{L_1}\right)^2,\ldots,\left(\frac{\pi}{L_p}\right)^2\right)$$
$$\left.\times\left[\gamma+\psi\left(\frac{q}{2}+1\right)\right]\right\} \qquad \text{for } q \text{ odd.} \tag{4.9}$$

## 5. Inhomogeneous multidimensional Epstein zeta functions

In this section we want to determine $\operatorname{Res} E_p^{c^2}(-N+\frac{1}{2}; a_1,\ldots,a_p)$, $O_p^{c^2}(-N+\frac{1}{2}; a_1,\ldots,a_p)$, $E_p^{c^2}(-N; a_1,\ldots,a_p)$ and $E_p'^{c^2}(-N; a_1,\ldots,a_p)$, $N\in\mathbb{N}_0$. Using regularization techniques for Mellin transforms, $\operatorname{Res} E_p^{c^2}(-N+\frac{1}{2}; a_1,\ldots,a_p)$ and $E_p^{c^2}(-N; a_1,\ldots,a_p)$ can be calculated in an elegant way [40], but in order to obtain all the needed quantities, the explicit analytical continuation of equation (4.2) to $\operatorname{Re}\nu<p/2$ has to be constructed.

Consider first $E_1^{c^2}(\nu; a)$. The analytical continuation is given by [10]

$$E_1^{c^2}(\nu;a)=-\frac{1}{2c^{2\nu}}+\sqrt{\frac{\pi}{a}}\frac{1}{2c^{2\nu-1}\Gamma(\nu)}$$
$$\times\left\{\Gamma\left(\nu-\frac{1}{2}\right)+4\sum_{l=1}^{\infty}\frac{a^{1/4-\nu/2}}{(\pi l c)^{1/2-\nu}}K_{1/2-\nu}\left(\frac{2\pi l c}{\sqrt{a}}\right)\right\} \tag{5.1}$$

with the Kelvin function $K_{1/2-\nu}$.

Here one can see, that the poles of the inhomogeneous Epstein zeta function are located at $\nu=-N+\frac{1}{2}$, $N\in\mathbb{N}$. This representation is very suitable to find the expansions

(4.4) and (4.5) for $p=1$,

$$E_1^{c^2}(\nu-N+\tfrac{1}{2};a)$$

$$=\frac{1}{\nu}\frac{1}{2}\sqrt{\frac{\pi}{a}}\frac{(-1)^N c^{2N}}{N!\Gamma(-N+\frac{1}{2})}-\frac{1}{2}c^{2N-1}$$
$$+\frac{1}{2}\sqrt{\frac{\pi}{a}}\frac{(-1)^N c^{2N}}{N!\Gamma(-N+\frac{1}{2})}[\psi(N+1)-\psi(N+\tfrac{1}{2})-\ln c^2]$$
$$+2\sqrt{\frac{\pi}{a}}\frac{c^{2N}}{\Gamma(-N+\frac{1}{2})}\sum_{l=1}^{\infty}\frac{a^{N/2}}{(\pi lc)^N}K_N\left(\frac{2\pi lc}{\sqrt{a}}\right)+\mathrm{O}(\nu) \tag{5.2}$$

$$E_1^{c^2}(\nu-N;a)$$

$$=-\frac{1}{2}c^{2N}+\nu\left\{\frac{1}{2}c^{2N}\ln c^2+\frac{1}{2}\sqrt{\frac{\pi}{a}}(-1)^N N!c^{2N+1}\right.$$
$$\left.\times\left[\Gamma(-N-\tfrac{1}{2})+4\sum_{l=1}^{\infty}\frac{a^{(N+\frac{1}{2})/2}}{(\pi lc)^{N+1/2}}K_{N+1/2}\left(\frac{2\pi lc}{\sqrt{a}}\right)\right]\right\}+\mathrm{O}(\nu^2). \tag{5.3}$$

Next we will express $E_p^{c^2}$ in terms of $E_1^{c^2}$, so using equations (5.2) and (5.3) the analogous expansions are obtained for $E_p^{c^2}$. First one finds (this recurrence relation corresponds to those given in [7], equation (4.9) and [9], equation (39)):

$$E_p^{c^2}(\nu;a_1,\ldots,a_p)$$

$$=-\frac{1}{2}E_{p-1}^{c^2}(\nu;a_1,\ldots,a_{p-1})+\frac{1}{2}\sqrt{\frac{\pi}{a_p}}\frac{\Gamma(\nu-\frac{1}{2})}{\Gamma(\nu)}E_{p-1}^{c^2}(\nu-\tfrac{1}{2};a_1,\ldots,a_{p-1})$$
$$+\frac{2}{\Gamma(\nu)}a_p^{-(\nu+1/2)/2}\pi^\nu\sum_{n_1,\ldots n_p=1}^{\infty}n_p^{\nu-1/2}[a_1n_1^2+\ldots+a_{p-1}n_{p-1}^2+c^2]^{(1/2-\nu)/2}$$
$$\times K_{1/2-\nu}\left(\frac{2\pi n_p}{\sqrt{a_p}}[a_1n_1^2+\ldots+a_{p-1}n_{p-1}^2+c^2]^{1/2}\right). \tag{5.4}$$

By induction it may then be shown that

$$E_p^{c^2}(\nu;a_1,\ldots,a_p)$$

$$=\frac{(-1)^{p-1}}{2^{p-1}}\frac{1}{\Gamma(\nu)}\sum_{k=0}^{p-1}(-1)^k\pi^{k/2}\Gamma\left(\nu-\frac{k}{2}\right)E_1^{c^2}(\nu-k/2;a_1)\sum_{\{j_1,\ldots,j_k\}}\prod_{r=1}^{k}\frac{1}{\sqrt{a_{j_r}}}$$
$$+\frac{\pi^\nu}{\Gamma(\nu)}\sum_{k=1}^{p-1}(-1)^{k+1}\sum_{\{j_1,\ldots,j_k\}}\mathscr{L}_k^{j_1}(\nu;a_1,\ldots,a_{j_1-1};c^2)$$
$$\times(-1)^{p+j_1}2^{j_1+1-p}\prod_{r=1}^{k}\frac{1}{\sqrt{a_{j_r}}} \tag{5.5}$$

where we define

$$\mathscr{L}_m^i(\nu;a_1,\ldots,a_{i-1};c^2)$$

$$=a_i^{-(\nu-m/2)/2}\sum_{n_1,\ldots,n_i=1}^{\infty}n_i^{\nu-m/2}[a_1n_1^2+\ldots+a_{i-1}n_{i-1}^2+c^2]^{(m/2-\nu)/2}$$
$$\times K_{m/2-\nu}\left(\frac{2\pi n_i}{\sqrt{a_i}}[a_1n_1^2+\ldots+a_{i-1}n_{i-1}^2+c^2]^{1/2}\right) \tag{5.6}$$

$\Sigma_{\{j_1,\ldots,j_k\}}$ denotes the sum over all possible choices of the $j_1<\ldots<j_k$ among $2,\ldots,p$.

Substituting equations (5.2) and (5.3) into equation (5.5), one obtains the expansions

$$
\begin{aligned}
E_p^{c^2}(\nu-N+\tfrac{1}{2};a_1,\ldots,a_p)
&=\frac{1}{\nu}\frac{(-1)^{p+N-1}}{2^p\Gamma(-N+\frac{1}{2})}\sum_{k=0}^{[(p-1)/2]}\frac{(-1)^k c^{2k+2N}}{(k+N)!}\pi^{k+1/2}B(2k+1)\\
&\quad+\frac{(-1)^p}{2^p}\sum_{k=0}^{[p/2]}\pi^k c^{2N+2k-1}\frac{\Gamma(-N-k+\frac{1}{2})}{\Gamma(-N+\frac{1}{2})}B(2k)\\
&\quad+\frac{(-1)^{p-1}}{2^p}\sum_{k=0}^{[(p-1)/2]}\frac{(-1)^{N+k}\pi^{k+1/2}c^{2N+2k}}{(k+N)!\Gamma(-N+\frac{1}{2})}B(2k+1)\\
&\quad\times[\psi(N+k+1)-\psi(N+\tfrac{1}{2})-\ln c^2]\\
&\quad+\frac{\pi^{-N+1/2}}{\Gamma(-N+\frac{1}{2})}\sum_{k=1}^{p}(-1)^{k+1}\sum_{\{i_1,\ldots i_k\}}\mathscr{L}_k^{i_1}(-N+\tfrac{1}{2};a_1,\ldots,a_{i_1-1};c^2)\\
&\quad\times(-1)^{p-i_1}2^{i_1+1-p}\prod_{r=1}^{k}\frac{1}{\sqrt{a_{i_r}}}+O(\nu)
\end{aligned}
\tag{5.7}
$$

$$
\begin{aligned}
E_p^{c^2}(\nu-N;a_1,\ldots,a_p)
&=\frac{(-1)^pN!}{2^p}\sum_{k=0}^{[p/2]}\frac{(-1)^k c^{2k+2N}\pi^k}{(k+N)!}B(2k)\\
&\quad+\nu\left\{\frac{(-1)^{N+p-1}N!}{2^p}\sum_{k=0}^{[(p-1)/2]}c^{2N+2k+1}\pi^{k+1/2}\Gamma(-N-k-\tfrac{1}{2})B(2k+1)\right.\\
&\quad+\frac{(-1)^{p-1}N!}{2^p}\sum_{k=0}^{[p/2]}\frac{(-1)^k\pi^k c^{2k+2N}}{(k+N)!}B(2k)[\ln c^2-\psi(N+k+1)+\psi(N+1)]\\
&\quad+(-1)^N N!\,\pi^{-N}\sum_{k=1}^{p}(-1)^{k+1}\sum_{\{i_1,\ldots,i_k\}}\mathscr{L}_k^{i_1}(-N;a_1,\ldots,a_{i_1-1};c^2)\\
&\quad\left.\times(-1)^{p-i_1}2^{i_1+1-p}\prod_{r=1}^{k}\frac{1}{\sqrt{a_{i_r}}}\right\}+O(\nu^2)
\end{aligned}
\tag{5.8}
$$

with

$$
B(l)=\sum_{\{i_1,\ldots,i_l\}}\frac{1}{\sqrt{a_{i_1}\ldots a_{i_l}}}. \tag{5.9}
$$

One can check, that with equations (3.8), (4.6), (4.7), (5.7) and (5.8) the result (4.3) of the general zeta function theory is recovered. Equations (5.7) and (5.8) are the main results of this section. Now all the quantities necessary to write down the high-temperature expansion (2.12) for the considered example are known.

## 6. High-temperature expansion for the example

In this section the results of the previous sections are collected to find the high-temperature expansion (2.12) for the specific configuration described in section 3.

First the contribution of the zeta function of the spatial section to the thermodynamic potential is determined to be

$$\zeta'_{n-1}(0,\mu)=\frac{L^q}{(4\pi)^{q/2}}\left\{\frac{(-1)^{p+1-\alpha}}{2^p}\sum_{k=0}^{[(p-1+\alpha)/2]}\pi^{-k-(1-\alpha)/2}\Gamma\left(-\frac{q+1-\alpha}{2}-k\right)\right.$$
$$\times(m^2-\mu^2)^{k+(q+1-\alpha)/2}A(2k+1-\alpha)$$
$$+\frac{(-1)^{p+(q-\alpha)/2}}{2^p}\sum_{k=0}^{[(p-\alpha)/2]}\frac{(-1)^k\pi^{-k-\alpha/2}}{(k+(q+\alpha)/2)!}(m^2-\mu^2)^{k+(q+\alpha)/2}A(2k+\alpha)$$
$$\times\left[\gamma+\psi\left(\frac{q+\alpha}{2}+k+1\right)-\ln(m^2-\mu^2)\right]$$
$$+\pi^{-q/2}\sum_{k=1}^{p}(-1)^{k+1}\pi^{-k}\sum_{\{i_1,\ldots,i_k\}}\mathcal{L}_k^{i_1}\left(-\frac{q}{2};\left(\frac{\pi}{L_1}\right)^2,\ldots,\left(\frac{\pi}{L_{i_1-1}}\right)^2;m^2-\mu^2\right)$$
$$\left.\times(-1)^{p-i_1}2^{i_1+1-p}\prod_{r=1}^{k}L_{i_r}\right\}\qquad\alpha=\begin{cases}0 & \text{for } q \text{ even}\\ 1 & \text{for } q \text{ odd.}\end{cases}\tag{6.1}$$

Next the coefficients $c_l$ in equation (2.12) are replaced by equation (3.8). Performing some elementary resummations, the thermodynamic potential may be cast into the form

$$\psi[\beta,\mu]=\frac{1}{(4\pi)^{n/2}}\left\{\ln\left(\frac{\beta}{4\pi\lambda}\right)A+A_0+\frac{2\pi}{\beta}\ln\left[\left(\frac{\beta}{2\pi}\right)^2(m^2-\mu^2)\right]\tilde{A}_{-1}+\frac{2\pi}{\beta}A_{-1}\right.$$
$$+\sum_{l=1}^{[(n-1)/2]}\left(\frac{\beta}{2\pi}\right)^{-2l-1}A_{-2l-1}+\sum_{l=1}^{[n/2]}\left(\frac{\beta}{2\pi}\right)^{-2l}A_{-2l}$$
$$\left.+\sum_{l=1}^{\infty}\left(\frac{\beta}{2\pi}\right)^{2l-1}A_{2l-1}+\sum_{l=1}^{\infty}\left(\frac{\beta}{2\pi}\right)^{2l}A_{2l}\right\}\tag{6.2}$$

with

$$A=2(-1)^{p+1+(q+\alpha)/2}\pi^{(p-1+\alpha)/2}$$
$$\times\sum_{k=0}^{[(p-1+\alpha)/2]}\frac{(-1)^k\pi^{-k}}{(k+1+(q-\alpha)/2)!}L^q m^{2k+q+2-\alpha}A(2k+1-\alpha)\tag{6.3}$$

$$A_0=-A\,\mathrm{Re}\,\psi(1+\mathrm{i}\rho)\tag{6.4}$$

$$\tilde{A}_{-1}=(-1)^{p+(q-\alpha)/2}\pi^{(p-1-\alpha)/2}$$
$$\times\sum_{k=0}^{[(p-\alpha)/2]}\frac{(-1)^k\pi^{-k}}{(k+(q+\alpha)/2)!}L^q(m^2-\mu^2)^{k+(q+\alpha)/2}A(2k+\alpha)\tag{6.5}$$

$$A_{-1}=2(-1)^{p+(q-\alpha)/2}\pi^{(p-1-\alpha)/2}\ln\left(\frac{2\sinh\pi\rho}{\rho}\right)$$
$$\times\sum_{k=0}^{[(p-\alpha)/2]}\frac{(-1)^k\pi^{-k}}{(k+(q+\alpha)/2)!}L^q m^{2k+q+\alpha}A(2k+\alpha)$$
$$+(-1)^{p+(q-\alpha)/2}\pi^{(p-1-\alpha)/2}\sum_{j=0}^{[(p-\alpha)/2]}\sum_{l=0}^{j+(q+\alpha)/2}\frac{(-1)^{j+l}\pi^{-j}\psi(1+l)}{l!(j+(q+\alpha)/2-l)!}$$

$$\times L^q m^{2j-2l+q+\alpha}\mu^{2l}A(2j+\alpha)$$
$$+(-1)^{p-\alpha}\pi^{(p+\alpha)/2-1}\sum_{k=0}^{[(p-1+\alpha)/2]}\pi^{-k}\Gamma\left(-k-\frac{q+1-\alpha}{2}\right)$$
$$\times A(2k+1-\alpha)L^q(m^2-\mu^2)^{k+(q+1-\alpha)/2}$$
$$+(-1)^{p+1+(q-\alpha)/2}\pi^{(p-1-\alpha)/2}\sum_{k=0}^{[(p-\alpha)/2]}\frac{(-1)^k\pi^{-k}}{(k+(q+\alpha)/2]!}\psi\left(\frac{q+\alpha}{2}+k+1\right)$$
$$\times L^q(m^2-\mu^2)^{k+(q+\alpha)/2}A(2k+\alpha)$$
$$+2(4\pi)^{(p-1)/2}\pi^{-q/2}\sum_{k=1}^{p}(-1)^k\pi^{-k}\sum\{i_1,\ldots,i_k\}$$
$$\times \mathscr{L}_k^{i_1}\left(-\frac{q}{2};\left(\frac{\pi}{L_1}\right)^2,\ldots,\left(\frac{\pi}{L_{i_1-1}}\right)^2;m^2-\mu^2\right)$$
$$\times(-1)^{p-i_1}2^{i_1+1-p}L^q\prod_{r=1}^{k}L_{i_r} \tag{6.6}$$

$$A_{-2l-1}=\frac{4(-1)^{p+(q-\alpha)/2+1}\pi^{(p-\alpha-1)/2}\operatorname{Re}\zeta'_{\mathrm{H}}(-2l,1+i\rho)}{l!}$$
$$\times\sum_{k=l-(q+\alpha)/2}^{[(p-\alpha)/2]}\frac{(-1)^k\pi^{-k}}{(k+(q+\alpha)/2-l)!}L^q m^{2k-2l+q+\alpha}A(2k+\alpha) \tag{6.7}$$

$$A_{-2l}=\frac{(-1)^{p+(q+\alpha)/2}\pi^{(p+\alpha)/2}\operatorname{Re}B_{2l}(\mathrm{i}\rho)}{l\Gamma(l+\frac{1}{2})}\sum_{k=l-(q+2-\alpha)/2}^{[(p-1+\alpha)/2]}\frac{(-1)^k\pi^{-k}}{(k+(q+2-\alpha)/2-l)!}$$
$$\times L^q m^{2k-2l+q+2-\alpha}A(2k+1-\alpha) \tag{6.8}$$

$$A_{2l-1}=2(-1)^{p+l+1+(q-\alpha)/2}\pi^{(p-1-\alpha)/2}\frac{(l-1)!}{(2l-1)!}\operatorname{Re}\psi^{(2l-1)}(1+\mathrm{i}\rho)$$
$$\times\sum_{k=0}^{[(p-\alpha)/2]}\frac{(-1)^k\pi^{-k}}{(k+l+(q+\alpha)/2)!}L^q m^{2k+2l+q+\alpha}A(2k+\alpha) \tag{6.9}$$

$$A_{2l}=2(-1)^{p+l+(q+\alpha)/2}\pi^{(p-2+\alpha)/2}\frac{\Gamma(l+\frac{1}{2})\operatorname{Re}\psi^{(2l)}(1+\mathrm{i}\rho)}{(2l)!}$$
$$\times\sum_{k=0}^{[(p-1+\alpha)/2]}\frac{(-1)^k\pi^{-k}}{(k+l+1+(q-\alpha)/2)!}L^q m^{2k+2l+q+2-\alpha}A(2k+1-\alpha). \tag{6.10}$$

Notice that the terms combine in such a way that the arguments of the logarithm terms are dimensionless, as required.

For the especially interesting case of two parallel plates at distance $L_1$ in three dimensions, (i.e. $p=1$, $q=2$) equations (6.2)–(6.10) simplify extremely and one obtains

$$A=-L^2L_1m^4 \tag{6.11}$$
$$A_0=\operatorname{Re}\psi(1+\mathrm{i}\rho)L^2L_1m^4 \tag{6.12}$$
$$\tilde{A}_{-1}=L^2(m^2-\mu^2) \tag{6.13}$$
$$A_{-1}=L^2\left\{2\ln\left(\frac{2\sinh\pi\rho}{\rho}\right)m^2-\tfrac{4}{3}(m^2-\mu^2)^{3/2}L_1-m^2\right.$$
$$\left.-\frac{4}{\sqrt{\pi}}L_1^{-1/2}(m^2-\mu^2)^{3/4}\sum_{n=1}^{\infty}n^{-3/2}K_{3/2}(2nL_1(m^2-\mu^2)^{1/2})\right\} \tag{6.14}$$

$$A_{-2}=2\,\mathrm{Re}\,B_2(\mathrm{i}\rho)L^2L_1m^2 \tag{6.15}$$

$$A_{-3}=-4\,\mathrm{Re}\,\zeta'_{\mathrm{H}}(-2,1+\mathrm{i}\rho)L^2 \tag{6.16}$$

$$A_{-4}=\tfrac{2}{3}\,\mathrm{Re}\,B_4(\mathrm{i}\rho)L^2L_1 \tag{6.17}$$

$$A_{2l-1}=\frac{2(-1)^{l+1}\,\mathrm{Re}\,\psi^{(2l-1)}(1+\mathrm{i}\rho)}{l(l+1)(2l-1)!}L^2m^{2l+2} \tag{6.18}$$

$$A_{2l}=\frac{2(-1)^l\Gamma(l+\frac{1}{2})\,\mathrm{Re}\,\psi^{(2l)}(1+\mathrm{i}\rho)}{\sqrt{\pi}(l+2)!(2l)!}L^2L_1m^{2l+4}. \tag{6.19}$$

In the limit of vanishing chemical potential (i.e. $\mu=0$ and $\rho=0$) one only has to replace the Bernoulli polynomials $B_{2l}(\mathrm{i}\rho)$ (respectively the Hyrwitz zeta function $\zeta'_{\mathrm{H}}(-2l,1+\mathrm{i}\rho)$) by the Bernoulli numbers $B_{2l}$ (respectively the Riemann zeta function $\zeta'_{\mathrm{R}}(-2l)$).

At the end of this section let us mention that $A_{-1}$ may be written in the form

$$\begin{aligned}A_{-1}=L^2\bigg\{&2\ln\left(\frac{2\sinh\pi\rho}{\rho}\right)m^2-\tfrac{4}{3}(m^2-\mu^2)^{3/2}L_1-m^2\\&-2\frac{(m^2-\mu^2)^{1/2}}{L_1}\mathrm{Li}_2\{\exp[-2L_1(m^2-\mu^2)^{1/2}]\}\\&-\frac{1}{L_1^2}\mathrm{Li}_3\{\exp[-2L_1(m^2-\mu^2)^{1/2}]\}\bigg\}\end{aligned} \tag{6.20}$$

where

$$\mathrm{Li}_N(x)=\sum_{n=1}^{\infty}\frac{1}{n^N}x^n$$

is the so-called polylogarithm function.

## 7. Thermodynamics of the Bose gas

Given the thermodynamic potential as a function of $\mu$, $L$, $L_i$ and $\beta$, all the physical quantities, for example, the particle density

$$\rho=-\frac{1}{L^q}\prod_{j=1}^{p}\frac{1}{L_j}\frac{\partial\psi[\beta,\mu]}{\partial\mu} \tag{7.1}$$

the energy

$$E=\left(\frac{\partial}{\partial\beta}-\frac{\mu}{\beta}\frac{\partial}{\partial\mu}\right)\beta\psi[\beta,\mu] \tag{7.2}$$

and the entropy

$$S=\beta^2\frac{\partial}{\partial\beta}\psi[\beta,\mu] \tag{7.3}$$

may be calculated.

Let us first consider the particle density $\rho$ of the system in order to examine the phenomenon of Bose–Einstein condensation. Some general remarks can be made using the results of [41, 42], where the ideal relativistic Bose gas (i.e. $p=0$) was considered. Bose–Einstein condensation of the system will take place for $q\geq 3$ at a relativistic temperature $T_c \gg m$ corresponding to $\mu_c = \pm(m^2+\sum_{j=1}^{p}(\pi/L_j)^2)^{1/2}$. Using equations (6.2), (6.7), (6.8), (7.1), the expansions of the Bernoulli polynomials [14] and the Hurwitz zeta function [43], one finds (where $\eta(s)=(1-2^{1-s})\zeta_R(s)$):

$$\rho = \mu\beta^{-n+2}\pi^{1-n/2}2^{3-n}(n-2)!\left\{\frac{n-1}{\pi^{5/2}\Gamma((n+1)/2)}\sum_{k=1}^{n/2}(-1)^k\frac{k\pi^{2k}}{(2k)!}\eta(n-2k)\right.$$
$$\left.-\frac{\beta}{2}\frac{\zeta_R(n-3)}{(n/2-1)!}\left(\sum_{j=1}^{p}\frac{1}{L_j}\right)\right\}+O(\beta^{-n+4}\mu) \qquad \text{for } n \text{ even} \tag{7.4}$$

and

$$\rho = \mu\beta^{-n+2}\pi^{(1-n)/2}2^{2-n}(n-1)!\left\{\frac{\zeta_R(n-2)}{((n-1)/2)!}\right.$$
$$-\frac{\beta}{\pi^{3/2}((n-1)/2)\Gamma(n/2)}\sum_{k=1}^{(n-1)/2}(-1)^k\frac{k\pi^{2k}}{(2k)!}\eta(n-1-2k)$$
$$\left.\times\left(\sum_{j=1}^{p}\frac{1}{L_j}\right)\right\}+O(\beta^{-n+4}\mu) \qquad \text{for } n \text{ odd} \tag{7.5}$$

which determines $T_c$. In the same way high-temperature expansions of the energy $E$ (7.2) and the entropy $S$ (7.3) may be found.

Let us now consider $q\leq 2$. For $q\leq 2$ the particle density diverges according to

$$-\frac{\mu}{2\pi\beta L_1}\ln(\mu_c^2-\mu^2) \qquad \text{for } q=2 \tag{7.6}$$

$$\frac{\mu}{\beta L_1 L_2}\frac{1}{(\mu_c^2-\mu^2)^{1/2}} \qquad \text{for } q=1 \tag{7.7}$$

$$\frac{2\mu}{\beta L_1 L_2 L_3}\frac{1}{\mu_c^2-\mu^2} \qquad \text{for } q=0 \tag{7.8}$$

as $\mu^2$ tends to $\mu_c^2$ and the system will not condense.

In order to find this behaviour, consider the expansion (6.2). At sight, it seems that the thermodynamic potential $\psi[\beta,\mu]$ has branch cuts from $\mu=m$ to $\mu=\infty$ and from $\mu=-m$ to $\mu=-\infty$, resulting from the zeta function of the spatial section. But looking at the definition (4.2) of the Epstein zeta function it is obvious, that the branch cuts begin at $\mu=\mu_c$, and in fact it is possible to analytically continue $\psi[\beta,\mu]$ to the region $m^2\leq\mu^2\leq\mu_c^2$, as will now be shown.

To explain the method, let us consider for simplicity $q=2$, $p=1$. The general case is also possible to handle, but needs more algebraic effort. Using the expansion of the polylogarithm function [12, 14], equation (6.20) may be written in the form

$$A_{-1} = L^2\left\{2m^2\ln\left(\frac{2\sinh\pi\rho}{\rho}\right)+4\sum_{l=1}^{\infty}\frac{(2L_1)^{2l}(m^2-\mu^2)^{l+1}(2l+1)}{(2l+2)!}\zeta_R(1-2l)\right.$$
$$\left.-2(m^2-\mu^2)\ln[2L_1(m^2-\mu^2)^{1/2}]-\mu^2-\frac{\zeta_R(3)}{L_1^2}\right\}$$

or, with the reflection formula for the zeta function of Riemann [27]

$$A_{-1}=L^2\left\{2m^2\ln\left(\frac{2\sinh\pi\rho}{\rho}\right)+2m^2-3\mu^2\right.$$
$$-2\left[\left(\frac{\pi}{L_1}\right)^2+m^2-\mu^2\right]\ln\left[1+\left(\frac{L_1}{\pi}\right)^2(m^2-\mu^2)\right]$$
$$-2(m^2-\mu^2)\ln(2L_1(m^2-\mu^2)^{1/2})-\frac{\zeta_R(3)}{L_1^2}$$
$$\left.+\frac{2\pi^2}{L_1^2}\sum_{l=1}^{\infty}\frac{(-1)^l[(L_1/\pi)^2(m^2-\mu^2)]^{l+1}}{l(l+1)}[\zeta_R(2l)-1]\right\}. \quad (7.9)$$

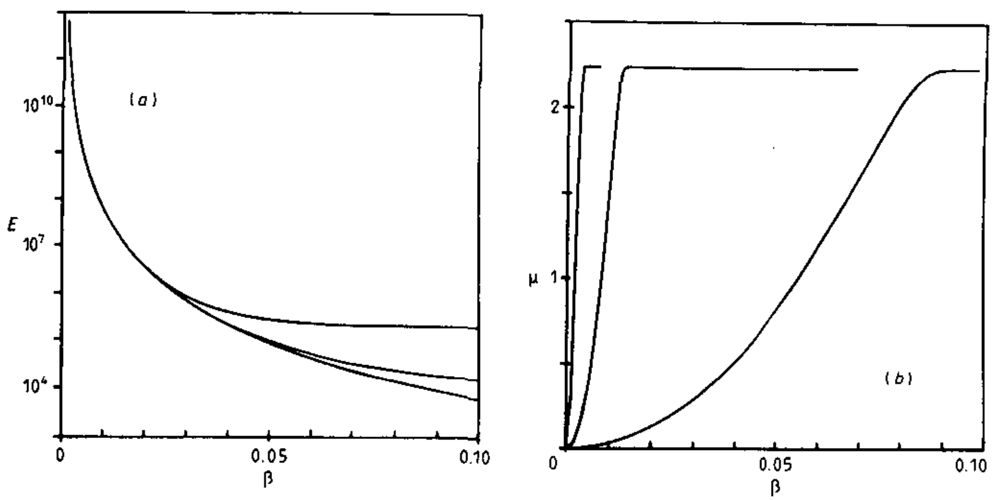


**Figure 1.** ($a$) $E$ and ($b$) $\mu$ are shown as a function of $\beta$ for $q=2$, $p=1$, $m=1$ and $L_1=0.5\pi$. The three curves in each diagram correspond to $\rho=100$, $5\times10^3$ and $10^5$ and $E$, $\mu$ are increasing with $\rho$.

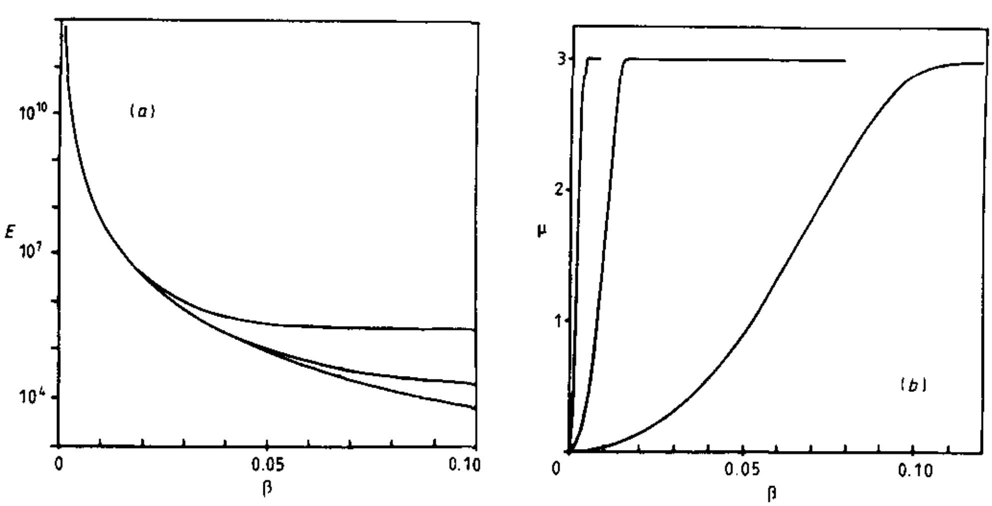


**Figure 2.** ($a$) $E$ and ($b$) $\mu$ are shown as a function of $\beta$ for $q=1$, $p=2$, $m=1$, $L_1=L_2=0.5\pi$.

This representation provides the analytical continuation of $\psi[\beta,\mu]$ to $m^2 \leqslant \mu^2 \leqslant \mu_c^2$. The leading orders are

$$\frac{\psi[\beta,\mu]}{L^2L_1} = -\frac{1}{45}\pi^2\beta^{-4} + \frac{1}{12}\beta^{-2}(m^2-2\mu^2) + \frac{1}{8\pi^2}\left[\frac{1}{3}\mu^4 - \frac{1}{2}m^4\ln\left(\frac{\beta}{4\pi\lambda}\right) - \frac{1}{2}\gamma m^4 - m^2\mu^2\right]$$
$$+\frac{1}{\pi L_1}\left\{\frac{1}{2}\zeta_R(3)\beta^{-3} + \beta^{-1}\left[\frac{1}{4}(m^2\ln\pi - \mu^2\ln 2\pi) + \frac{1}{4}(m^2-\mu^2)\ln\left(\frac{\beta}{4\pi L_1}\right)\right.\right.$$
$$-\frac{1}{8}\frac{\zeta_R(3)}{L_1^2} + \frac{1}{4}m^2 + \frac{\pi^2}{4L_1^2}\sum_{l=1}^{\infty}\frac{(-1)^l}{l(l+1)}\left[\left(\frac{L_1}{\pi}\right)^2(m^2-\mu^2)\right]^{l+1}[\zeta_R(2l)-1]$$
$$\left.\left.-\frac{1}{4}\left[\left(\frac{\pi}{L_1}\right)^2 + m^2 - \mu^2\right]\ln\left(1+\left(\frac{L_1}{\pi}\right)^2(m^2-\mu^2)\right)\right]\right\} + O(\beta). \qquad (7.10)$$

Obviously the thermodynamic potential has branch cuts from $\mu=(m^2+(\pi/L_1)^2)^{1/2}$ to $\mu=\infty$ and from $\mu=-(m^2+(\pi/L_1)^2)^{1/2}$ to $\mu=-\infty$, as it should, and the behaviour (7.6) is found. It is now easy to derive the high-temperature expansion of $\rho$, $E$ and $S$ valid for $|\mu|\leqslant|\mu_c|$.

In order to examine, for example, $E$ as a function of $\rho$ and $\beta$, note first that the expansion of $\rho$ is really an implicit formula for $\mu$ as a function of $\rho$ and $\beta$. This is solved numerically (we use the first five orders in $\beta$) and the results shown in figures 1–3 for $q\leqslant 2$ are obtained. For completeness let us mention, that using the low-temperature expansion of the thermodynamic potential (which may be found simply by Poisson resummation) the scaling length $\lambda$ has been determined in this analysis in such a way that the vacuum zero-temperature energy vanishes.

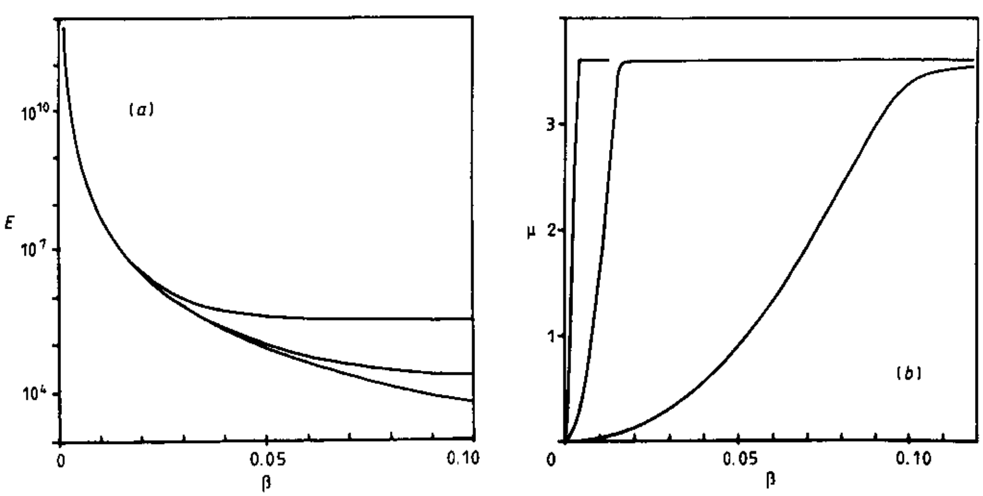


**Figure 3.** (*a*) $E$ and (*b*) $\mu$ are shown as a function of $\beta$ for $q=0$, $p=3$, $m=1$, $L_1=L_2=L_3=0.5\pi$.

## 8. Conclusions

Using zeta function regularization and heat-kernel techniques a high-temperature expansion of the thermodynamic potential in an arbitrary ultrastatic spacetime with boundary has been derived, equation (2.12). Based on new results concerning the Minakshisundaram–Pleijel coefficient $c_2$ [21, 44–46], this expansion is known explicitly up to order $\beta^{-n+4}$ for $n \neq 4$ and $\ln \beta$ for $n=4$.

It is shown that equation (2.12) is very suitable for calculating the expansion for specific configurations, i.e. we discussed a massive bosonic field subject to Dirichlet boundary conditions on hypercuboids in a flat $n$-dimensional spacetime (6.2). The example involves the analysis of multidimensional inhomogeneous Epstein zeta functions. The essential results, interesting also from a pure mathematical point of view, are equations (5.7) and (5.8). The thermodynamics of the system has been considered, i.e. a numerical analysis of some thermodynamical quantities has been given.

Other boundary value problems can be treated by slightly modifying the considerations of sections 3–6. Furthermore, an analogous approach for higher-spin particles may be developed.

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